Journal of Engineering Mathematics **44:** 245–257, 2002.

Bifurcation of cavitation solutions for incompressible transversely isotropic hyper-elastic materials

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Received 9 October 2001; accepted in revised form 30 July 2002

Abstract. In this paper, the bifurcation problem of void formation and growth in a solid circular cylinder, composed of an incompressible, transversely isotropic hyper-elastic material, under a uniform radial tensile boundary dead load and an axial stretch is examined. At first, the deformation of the cylinder, containing an undetermined parameter-the void radius, is given by using the condition of incompressibility of the material. Then the exact analytic formulas to determine the critical load and the bifurcation values for the parameter are obtained by solving the differential equation for the deformation function. Thus, an analytic solution for bifurcation problems in incompressible anisotropic hyper-elastic materials is obtained. The solution depends on the degree of anisotropy of the material. It shows that the bifurcation may occur locally to the right or to the left, depending on the degree of anisotropy, and the condition for the bifurcation to the right or to the left is discussed. The stress distributions subsequent to the cavitation are given and the jumping and concentration of stresses are discussed. The stability of solutions is discussed through comparison of the associated potential energies. The bifurcation to the left is a 'snap cavitation'. The growth of a pre-existing void in the cylinder is also observed. The results for a similar problem in three dimensions were obtained by Polignone and Horgan.

Key words: bifurcation, comparison of energy, incompressible hyper-elastic material, jumping and concentration of stress, transversely isotropic cylinder

1. Introduction

Void formation and growth in solid materials due to the instability of materials play a fundamental role in the mechanisms of fracture and failure of materials. In recent years, macromolecular materials such as polyurethane are achieving a more and more important status in materials research and used in almost all fields of modern science. This is why nonlinear problems of void formation and growth in hyper-elastic materials have attracted much attention.

In 1958, Gent and Lindley [1] observed the sudden formation of voids in hyper-elastic materials in their experimental work on rubber cylinders. In 1982, Ball [2] modelled the sudden formation of a void in hyper-elastic materials (once a critical load is attained) as a class of bifurcation problems in the nonlinear theoretical investigation in solid mechanics. Bifurcation problems of void formation and growth, both for incompressible and compressible materials, were carried out. For general incompressible hyper-elastic materials an explicit formula to determine the critical load was given and for compressible hyper-elastic materials, a qualitative analyses was given. Ball [2], Horgan and Abeyaratne [3], Sivaloganathan [4] gave

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an alternative interpretation of cavitation in terms of the sudden rapid growth of a pre-existing micro-void.

Chou-Wang and Horgan [5] studied void nucleation and growth for some kinds of incompressible materials. For hyperelastic materials, cavitation solutions do not always exist. For example, cavitation cannot occur in a Mooney-Rivlin material. Horgan and Abeyaratne [3], Sivaloganathan [4], Stuart [6], Meynard [7], Horgan [8], Shang and Cheng [9, 10], Podio-Guidugli *et al.* [11] and Hao [12] studied the bifurcation problem for compressible hyper-elastic materials. Qualitative analyses regarding existence, uniqueness and stability of cavitation solutions were established and some results given. In a recent paper [13], Horgan and Polignone reviewed bifurcation problems for radially symmetric cavitation for hyper-elastic materials, which may be homogeneous and isotropic or inhomogeneous and anisotropic, incompressible or compressible. It also gives an extensive list of references.

For other aspects of cavitation problems, Horgan and Pence [14, 15] studied the effects of material inhomogeneity on bifurcation problems for incompressible materials. They obtained cavitation solutions for a sphere composed of two materials and showed bifurcation to occur locally either to the right or to the left. Sivaloganathan [16], Antman and Negron-Marrero [17], Polignone and Horgan [18, 19] studied the effect of material anisotropy on the formation and growth of voids for incompressible hyper-elastic materials. In [18], the authors obtained an analytic solution for a transverse isotropic sphere composed of a neo-Hookean material and showed bifurcation occurs either to the right or to the left. Generalization to the case of composite anisotropic materials was given in [19].

The purpose of the present paper is to further investigate the bifurcation problem for the formation and growth of voids for incompressible anisotropic hyper-elastic materials. The bifurcation problem for a circular cylinder, composed of an incompressible anisotropic Ogden material with transverse isotropy about the radial direction, under a uniform radial tensile dead-load and axial stretch is investigated. From the condition of incompressibility of the material, the radially symmetric deformation function of the problem is given by means of an undetermined parameter describing the growth of the cavity.

An exact analytic relation between the parameter, the load, as well as an explicit formula to determine the critical load are obtained by solving the differential equation satisfied by the deformation function. Thus, a new analytic solution for the bifurcation problem of void formation and growth for an incompressible anisotropic hyper-elastic material is obtained following the work of Polignone and Horgan [18]. The solution depends on the degree of anisotropy of the material. When the load exceeds a critical value, there exist homogeneous and cavitation solutions and the cavitation solutions bifurcate from the homogeneous solution at the critical load.

The relations between the critical load and the axial stretch, the degree of anisotropy of the material and the material parameter are discussed in detail. The bifurcation curves are obtained from numerical calculations based on the analytic solution. Different from the result obtained for isotropic materials, the bifurcation may here occur locally to the right or to the left and the relevant condition depends on the degree of anisotropy of the material. A similar result was presented by Polignone and Horgan [18].

The stress distributions subsequent to the cavitation are analyzed, and the jumping and concentration of stresses as well as the effect of the degree of anisotropy on the stress distributions are all observed. The stability of solutions is discussed through a comparison of associated potential energies. The potential energy associated with the cavitation solution is always lower than that associated with the homogeneous solution for the bifurcation to the right, so the cavitation solution is stable and the radius of the void is continuously increasing from zero. The potential energy associated with the cavitation solution for the bifurcation to the left is more complex; a cavity with a finite radius may suddenly appear at the critical load and so the bifurcation to the left is a 'snap cavitation'.

The condition for the bifurcation to the right or to the left is discussed from the bifurcation curves and the energy curves. It is shown that the bifurcation condition depends on the degree of anisotropy of the material and there exists a critical value for the degree of anisotropy. When the degree of anisotropy is less than the critical value, the bifurcation is to the right and when the degree of anisotropy is larger than the critical value, the bifurcation is to the left.

Finally, the sudden growth of a pre-existing void in the center of the cylinder is observed. The growth character of the pre-existing voids also depends on the degree of anisotropy of the material. When the degree of anisotropy is less than the critical value, the growth of the void is continuous and when the degree of anisotropy is larger than the critical value, the growth of the void is discontinuous with a 'jump'.

2. Formulation for the problem

Consider here the finite deformation of a solid circular cylinder with radius b, composed of an incompressible anisotropic Ogden material with a transversely isotropy about the radial direction. Assume that the cylinder is subjected to a uniform radial tensile dead-load p_0 on its boundary surface $R = b$ and an axial stretch or compression λ_3. The undeformed and the deformed configurations are described by the cylindrical coordinate systems (R, Θ, Z) and (r, θ, z), with the origin at the center of the cylinder, respectively. Assume that the deformation function of the cylinder is radially symmetric, namely,

$$r = r(R) > 0, \quad \theta = \Theta, \quad z = \lambda_3 Z, \tag{1}$$

where $r(R)$ is an undetermined function. The deformation gradient tensor is given as

$$\boldsymbol{F} = \text{diag}\,(\dot{r}(R), r(R)/R, \lambda_3) = \text{diag}(\lambda_R, \lambda_\Theta, \lambda_Z) \tag{2}$$

in which, $\lambda_R, \lambda_\Theta, \lambda_z$ are the principal stretches given by

$$\lambda_R = \dot{r}(R) = \frac{\mathrm{d}r}{\mathrm{d}R}, \quad \lambda_\Theta = \frac{r(R)}{R}, \quad \lambda_Z = \lambda_3. \tag{3}$$

The corresponding right and left Green-Cauchy deformation tensors are

$$\boldsymbol{C} = \boldsymbol{B} = \text{diag}\left(\dot{r}^2(R), r^2(R)\big/R^2, \lambda_3^2\right). \tag{4}$$

The strain-energy function of the incompressible anisotropic Ogden material with a transverse isotropy about the radial direction is given as [18, 20, 21]

$$W = \frac{\mu}{\alpha}\left[\left(\lambda_R^\alpha + \lambda_\Theta^\alpha + \lambda_Z^\alpha - 3\right) + af(I_4, I_5)\right] \tag{5}$$

in which μ and α are material parameters; a is a dimensionless parameter measuring the degree of anisotropy of the material and when $a = 0$, the material is isotropic. The invariants I_4 and I_5 are given by

$$I_4 = C_{12}^2 + C_{13}^2, \quad I_5 = C_{11}. \tag{6}$$

It is easy to obtain $I_4 \equiv 0$ from the deformation tensor (4). So the function f in (5) only depends on I_5, that is, $f = f(I_5)$. From the normalization condition for the strain energy function $W = W(\lambda_R, \lambda_\Theta, \lambda_Z, I_4, I_5)$, we have [18]

$$W(1,1,1,0,1) = 0, \quad \frac{\partial W}{\partial I_5}(1,1,1,0,1) = 0.$$

We may now choose the representation for $f(I_5)$ as $f(I_5) = I_5^3 - 3I_5 + 2$ and, in this case, (5) may be rewritten as

$$W = \frac{\mu}{\alpha}\left[\left(\lambda_R^\alpha + \lambda_\Theta^\alpha + \lambda_Z^\alpha - 3\right) + a\left(\lambda_R^6 - 3\lambda_R^2 + 2\right)\right]. \tag{7}$$

The corresponding Cauchy-stress components are

$$\tau_{rr}(R) = \lambda_R \frac{\partial W}{\partial \lambda_R} - p(R), \quad \tau_{\theta\theta}(R) = \lambda_\Theta \frac{\partial W}{\partial \lambda_\Theta} - p(R), \quad \tau_{zz}(R) = \lambda_Z \frac{\partial W}{\partial \lambda_Z} - p(R) \tag{8}$$

in which $p(R)$ is the undetermined hydrostatic pressure.

The equilibrium equation for the cylinder in the absence of body forces is

$$\frac{d\tau_{rr}}{dR} + \frac{\dot{r}(R)}{r(R)}[\tau_{rr} - \tau_{\theta\theta}] = 0. \tag{9}$$

The boundary condition at the outer edge $(R = b)$ is

$$\tau_{rr}(b) = p_0\left[\frac{b}{\lambda_3 r(b)}\right], \tag{10}$$

where $p_0 > 0$ is the prescribed dead-load and $\lambda_3 > 0$ is the prescribed axial stretch or compression. When a cavity forms, the boundary condition of the cavity surface should be

$$\lim_{R\to 0^+} r(R) = c > 0, \quad \lim_{R\to 0^+} \tau_{rr}(R) = 0. \tag{11}$$

Now, the problem is to seek the deformation function (1) and stress components (8), so that the equilibrium equation (9), the boundary conditions (10) and (11) are all satisfied for the given dead-load $p_0 > 0$ and axial stretch or compression $\lambda_3 > 0$ with a given strain-energy function (7).

3. Analytic solution

From the incompressibility condition of the material, namely, $J = \mathrm{Det}\boldsymbol{F} = 1$, and from (2), we have

$$\lambda_3 \dot{r}(R)\frac{r(R)}{R} = 1. \tag{12}$$

Integrating (12), we have

$$r(R) = \left(\frac{R^2}{\lambda_3} + c^2\right)^{1/2}, \tag{13}$$

where c (the radius of the void) is an undetermined parameter. Let

$$v = v(R) = \lambda_\Theta = \frac{r(R)}{R} = \left(\frac{1}{\lambda_3} + \frac{c^2}{R^2}\right)^{1/2}. \tag{14}$$

We have

$$\dot{r}(R) = \lambda_R = \lambda_3^{-1} v^{-1}. \tag{15}$$

So the strain-energy function (7) can be written as

$$W = \frac{\mu}{\alpha}\left[\lambda_3^{-\alpha} v^{-\alpha} + v^{\alpha} + \lambda_3^{\alpha} + 3) + a(\lambda_3^{-6} v^{-6} - 3\lambda_3^{-2} v^{-2} + 2)\right]. \tag{16}$$

The Cauchy-stress components are now given as

$$\begin{aligned} \tau_{rr}(R) = {} & \mu\lambda_3^{-\alpha} v^{-\alpha} + 6\mu a\alpha^{-1}\lambda_3^{-6} v^{-6} - 6\mu a\alpha^{-1}\lambda_3^{-2} v^{-2} - p(R), \\ \tau_{\theta\theta}(R) = {} & \mu v^{\alpha} - p(R), \quad \tau_{zz}(R) = \mu\lambda_3^{\alpha} - p(R). \end{aligned} \tag{17}$$

Thus, the problem reduces to seeking the pressure function $p(R)$ and a parameter c, for a given dead-load $p_0 > 0$ and axial stretch or compression $\lambda_3 > 0$, so that the equilibrium equation (9), the boundary conditions (10) and (11) are all satisfied. If $c > 0$, a void will form at the center of the solid cylinder and, if $c = 0$, the solid cylinder remains solid.

It is easy to show that the problem always has a trivial solution for all values of p_0 and λ_3, that is,

$$p(R) = \mu - p_0, \quad c = 0. \tag{18}$$

The trivial solution corresponds to the identity deformation states, namely, $r(R) = R$. So the cylinder retains its undeformed state, but there is a homogeneous stress state, that is, $\tau_{rr} = \tau_{\theta\theta} = \tau_{zz} = p_0$.

In order to seek the cavitation solution for $c > 0$, substituting (17) in (9) and using the variable transformation $r(R) = Rv(R)$, we may rewrite the equilibrium equation as

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[\mu\lambda_3^{-\alpha} v^{-\alpha} + 6\mu a\alpha^{-1}\lambda_3^{-6} v^{-6} - 6\mu a\alpha^{-1}\lambda_3^{-2} v^{-2} - p(R)\right]$$
$$+\frac{v^{-2}}{\lambda_3 R}(\mu\lambda_3^{-\alpha} v^{-\alpha} + 6\mu a\alpha^{-1}\lambda_3^{-6} v^{-6} - 6\mu a\alpha^{-1}\lambda_3^{-2} v^{-2} - \mu v^{\alpha}) = 0.$$

Integration yields

$$p(R) - p(0) = \mu\lambda_3^{-\alpha} v^{-\alpha} + 6\mu a\alpha^{-1}\lambda_3^{-6} v^{-6} - 6\mu a\alpha^{-1}\lambda_3^{-2} v^{-2} + J(R), \tag{19}$$

where

$$J(R) = \mu\lambda_3^{-1}\int_0^R \left(\lambda_3^{-\alpha} v^{-\alpha-2} + 6a\alpha^{-1}\lambda_3^{-6} v^{-8} - 6a\alpha^{-1}\lambda_3^{-2} v^{-4} - v^{\alpha-2}\right)\frac{\mathrm{d}s}{s}. \tag{20}$$

Substituting (19) in (17), we obtain the stress component

$$\tau_{rr}(R) = -p(0) - J(R). \tag{21}$$

Using (21) and $J(0) = 0$ and (11), we obtain $p(0) = 0$. Finally, from (21) and the boundary condition (10), we have

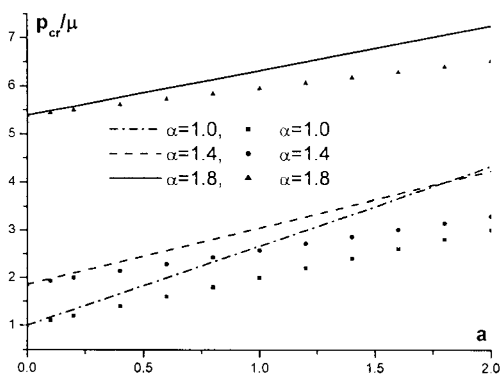

Figure 1. $p_{\rm cr} \sim a$ curves.

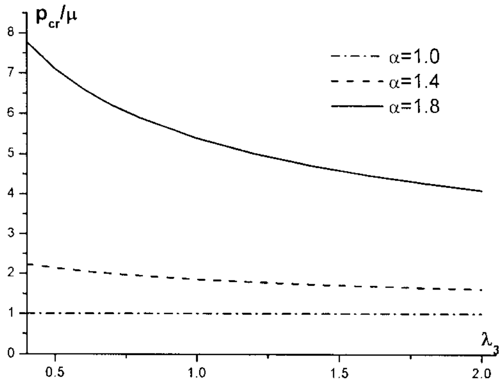

Figure 2. $p_{\rm cr} \sim \lambda_3$ curves.

$$p_0 = -\lambda_3 \nu(b) J(b). \tag{22}$$

Using (14) and observing its derivative relation $\frac{\mathrm{d}R}{R} = \frac{\nu}{\lambda_3^{-1} - \nu^2}\mathrm{d}\nu$, we arrive at

$$p_0 = \mu \left(\frac{1}{\lambda_3} + \frac{c^2}{b^2}\right)^{1/2} \left[\int_{\left(\frac{1}{\lambda_3}+\frac{c^2}{b^2}\right)^{1/2}}^{\infty} \frac{\lambda_3^{-\alpha}\nu^{-1-\alpha} + 6a\alpha^{-1}\lambda_3^{-6}\nu^{-7} - 6a\alpha^{-1}\lambda_3^{-2}\nu^{-3} - \nu^{\alpha-1}}{\lambda_3^{-1} - \nu^2}\mathrm{d}\nu \right]. \tag{23}$$

Expression (23) is an exact analytic relation between the cavity radius c and the applied dead-load p_0 and axial stretch or compression λ_3. One can see that the solution not only depends on the material parameters μ and α, as well as the geometric dimension b of the cylinder, but also on the parameter a measuring the degree of anisotropy of the material. For a given dead-load p_0 and axial stretch or compression λ_3, the parameter c corresponding to various values of a and α may be obtained from (23). If there exists a positive root $c > 0$ for (23), this means that a void forms in the solid cylinder.

The corresponding principal stresses are

$$\begin{aligned} \tau_{rr}(R) &= -J(R), \\ \tau_{\theta\theta}(R) &= \mu\nu^{\alpha} - \mu\lambda_3^{-\alpha}\nu^{-\alpha} - 6\mu a\alpha^{-1}\lambda_3^{-6}\nu^{-6} + 6\mu a\alpha^{-1}\lambda_3^{-2}\nu^{-2} + \tau_{rr}(R). \end{aligned} \tag{24}$$

Letting $c \to 0^+$ in (23), we observe that the critical load $p_{\rm cr}$ at which an internal void will form is given as

$$p_{\rm cr} = \mu\sqrt{\frac{1}{\lambda_3}} \int_{\sqrt{\frac{1}{\lambda_3}}}^{\infty} \frac{\lambda_3^{-\alpha}\nu^{-1-\alpha} + 6a\alpha^{-1}\lambda_3^{-6}\nu^{-7} - 6a\alpha^{-1}\lambda_3^{-2}\nu^{-3} - \nu^{\alpha-1}}{\lambda_3^{-1} - \nu^2}\mathrm{d}\nu. \tag{25}$$

Numerical evaluation of (25) yields the critical-load value $p_{\rm cr}$ for given values of the parameters a and α. In Figure 1 and Figure 2, the curves $p_{\rm cr}$ *vs.* a and $p_{\rm cr}$ *vs.* λ_3 for various values

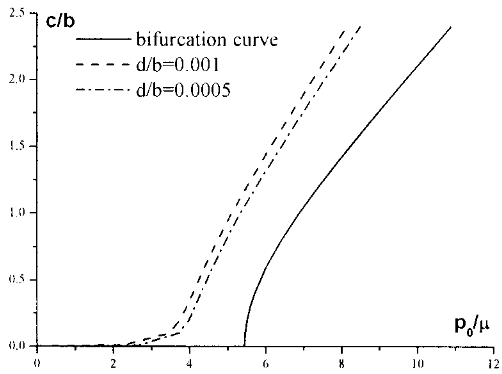

Figure 3. Bifurcation and growth curves for $\alpha = 0{\cdot}1$ and $\alpha = 1{\cdot}8$.

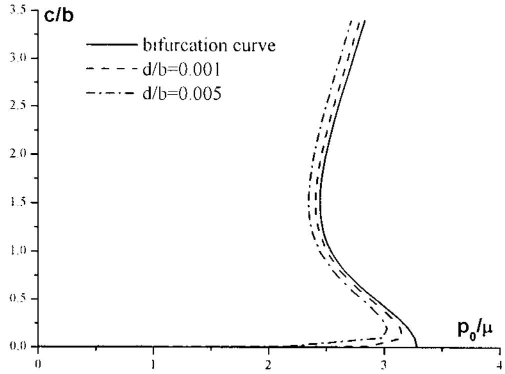

Figure 4. Bifurcation and growth curves for $\alpha = 2{\cdot}$ and $\alpha = 1{\cdot}4$.

of α are shown. To ensure the existence of p_{cr}, the material parameter α should be less than 2 (In fact, if $\alpha \geq 2$, the value of p_{cr} will be infinite.). It shows that the critical load p_{cr} increases with increasing a and the minimum is attained at $a = 0$, that is, the material is isotropic. It also shows that the critical load p_{cr} increases with increasing α but decreases with increasing λ_3. That is to say, an axial stretch $\lambda_3 > 1$ leads to the formation of a void under a radial tensile dead-load.

4. Bifurcation and stress distribution

If $p_0 < p_{\text{cr}}$, there is a unique solution of (23), that is, $c = 0$, so the problem only has the trivial solution. If $p_0 \geq p_{\text{cr}}$, there is also a solution of (23) with $c > 0$, apart from $c = 0$. So there exist cavitation solutions bifurcating from the trivial solution at the critical value p_{cr}. The subsequent growth of the cavity can also be found from (23). In the cases of $a = 0{\cdot}1$ and $\alpha = 0{\cdot}8$ and $a = 2{\cdot}0$ and $\alpha = 1{\cdot}4$ (assuming $\lambda_3 = 1$ for convenience), the bifurcation curves obtained from (23) are shown in Figures 3 and 4, respectively. In the case shown in Figure 3, the bifurcation is locally to the right and in the case shown Figure 4, it is locally to the left. It can be seen from Figures 3 and 4 that the cavity in the cylinder suddenly appears, once the load p_0 attains its critical value p_{cr} and subsequently the cavity grows rapidly. One can also see that the bifurcation may here occur locally either to the right or to the left, depending on the degree of anisotropy of the material. This is different from the situation for isotropic materials (see [18]).

Curves for the radial displacement $u(R) = r(R) - R$ for different values of p_0 are shown in Figure 5 when $\lambda_3 = 1$. When $p_0 < p_{\text{cr}}$, though the cylinder undergoes a homogeneous stress state, it remains undeformed. When $p_0 \geq p_{\text{cr}}$, a cavitating deformation bifurcates from the undeformed state. The displacement of the cylinder decreases with increasing R.

When $p_0 \geq p_{\text{cr}}$, the principal stresses obtained from (24) are shown in Figures 6 and 7 for $a = 0{\cdot}1$ and $\alpha = 1{\cdot}8$, respectively. It can be seen that the radial stress τ_{rr} is zero at the cavity surface and increases rapidly with increasing radius R and approaches an asymptotic value in the region far from the cavity. On the other hand, the circumferential stress $\tau_{\theta\theta}$ is infinite at the cavity surface and decreases rapidly with increasing R and approaches an asymptotic value in

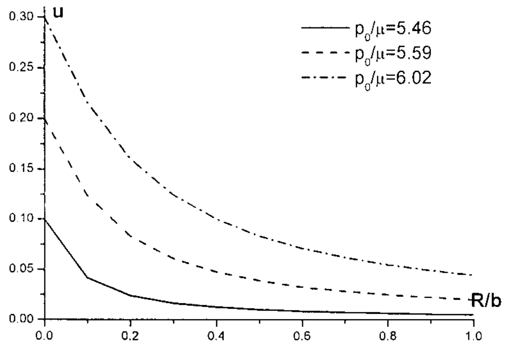

Figure 5. $u \sim \frac{r}{b}$ curves for $\alpha = 0{\cdot}1$ and $\alpha = 1{\cdot}8$.

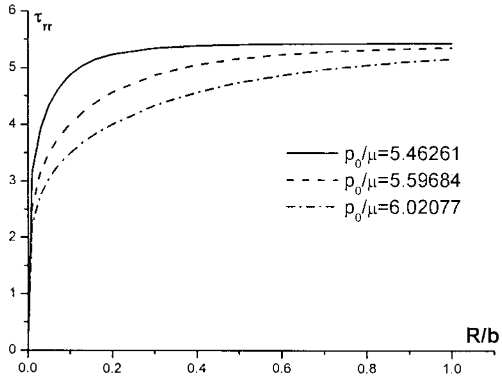

Figure 6. $\tau_{rr} \sim \frac{R}{b}$ curves for $\alpha = 0{\cdot}1$ and $\alpha = 1{\cdot}8$.

the region far from the cavity. At the same time, τ_{rr} decreases with increasing dead-load p_0, but $\tau_{\theta\theta}$ increases with increasing dead-load p_0. The asymptotic value of τ_{rr} may be close to the asymptotic value of $\tau_{\theta\theta}$ in the region far from the cavity, but the inequality $\tau_{\theta\theta}(R) > \tau_{rr}(R)$ is always true for $c > 0$.

The stresses corresponding to the trivial solution are given by the homogeneous state of stress $\tau_{rr} = \tau_{\theta\theta} = \tau_{zz} = p_0$ when $p_0 < p_{\mathrm{cr}}$. Thus, when a cavity is formed at $p_0 = p_{\mathrm{cr}}$, the stresses undergo an obviously catastrophic transition from the homogeneous distribution to the non-homogeneous distribution. In fact, as shown in Figure 8, the radial stress τ_{rr} and the circumferential stress $\tau_{\theta\theta}$ jump from p_0 to zero and from p_0 to infinity at the surface of the cavity, respectively. From the figures, one can also see that there obviously exist the stress concentration phenomenon for the circumferential stress $\tau_{\theta\theta}$ near the region of the cavity. As shown in Figure 9, the concentration factor of $\tau_{\theta\theta}$ is infinite at the surface of the cavity and decreases rapidly with increasing R and is close to an asymptotic value, slightly larger than 1·0, in the region far from the cavity. This implies that the stress concentration is a local phenomenon, but this is just the reason for the sudden appearance of the cavity and its subsequent rapid growth.

5. Comparison of potential energies

From the above analysis, one can see that, in the case of $a = 0{\cdot}1$ and $\alpha = 1{\cdot}8$, the cavitated bifurcation occurs locally to the right when $p_0 \geq p_{\mathrm{cr}}$, so there are two equilibrium solutions. For $a = 2{\cdot}0$ and $\alpha = 1{\cdot}4$, the cavitated bifurcation occurs locally to the left, so there are three equilibrium solutions when $p_n < p_0 < p_{\mathrm{cr}}$ (in which (c_n, p_n) is the knee on the bifurcation curve shown in Figure 12) but there are two equilibrium solutions when $p_0 \geq p_{\mathrm{cr}}$. In order to determine the stability of the solutions, including the trivial and the cavitation solutions, it is necessary to compute and compare the potential energies corresponding to the solutions. The total potential energy of the cylinder subjected to a dead-load p_0 is given as

$$E = \int_V W \mathrm{d}V - \int_A p_0(r(b) - b)\mathrm{d}A = 2\pi \int_0^b RW\mathrm{d}R - 2\pi b p_0(r(b) - b). \qquad (26)$$

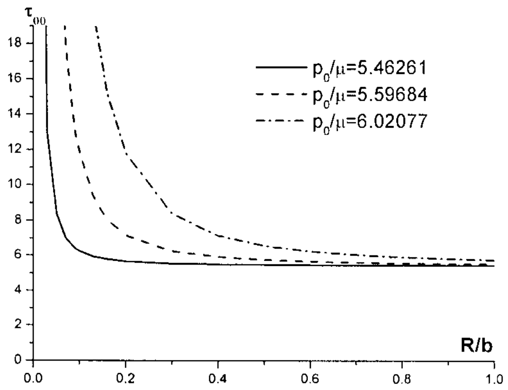

Figure 7. $\tau_{\theta\theta} \sim \frac{R}{b}$ curves for $\alpha = 0{\cdot}1$ and $\alpha = 1{\cdot}8$.

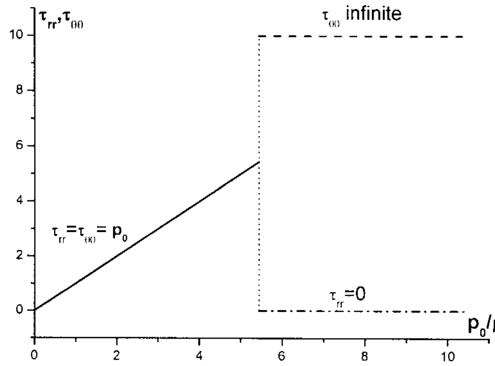

Figure 8. Stress jumping at cavity surface.

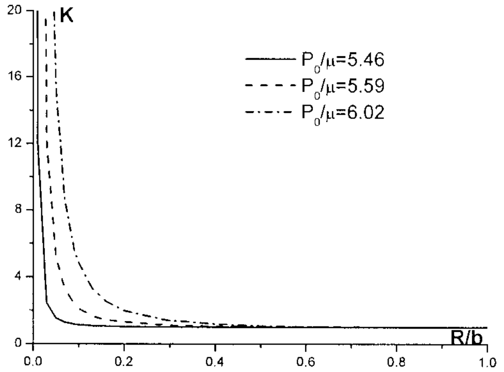

Figure 9. $K \sim \frac{R}{b}$ curves.

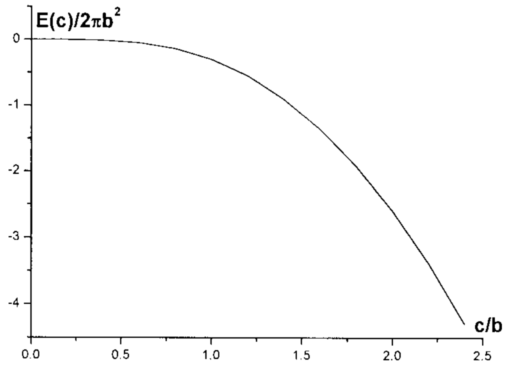

Figure 10. Energy curve for $a = {\cdot}1$ and $\alpha = 1{\cdot}8$.

It is clear that the potential energy for the trivial solution is equal to zero, namely, $E(0) = 0$. For the cavitation solution, the potential energy is given as

$$E(c) = 2\pi c^2 \int_{\left(\frac{1}{\lambda_3}+\frac{c^2}{b^2}\right)^{1/2}}^{\infty} \left[\frac{\mu}{\alpha}(\lambda_3^{-\alpha}\nu^{-\alpha+1} + \nu^{\alpha+1} + \lambda_3^{\alpha}\nu - 3\nu) + a(\lambda_3^{-6}\nu^{-5} - 3\lambda_3^{-2}\nu^{-1} + 2\nu)\right] \frac{1}{(\lambda_3^{-1} - \nu^2)^2} \mathrm{d}\nu - 2\pi b^2 p_0 \left[\left(\frac{1}{\lambda_3} + \frac{c^2}{b^2}\right)^{1/2} - 1\right]. \tag{27}$$

For $a = 0{\cdot}1$ and $\alpha = 1{\cdot}8$ or $a = 2{\cdot}0$ and $\alpha = 1{\cdot}4$, the numerical results obtained from (27) are shown in Figures 10 and 11, respectively. One can see from Figure 10, that $E(c)$ is always less than $E(0)$, so the cavitation solution is stable when $p_0 \geq p_{\mathrm{cr}}$. From Figure 11, $E(c)$ is larger than $E(0)$ for $0 < c < c_n$ and is a monotonically increasing function of c which attains its maximum at $c = c_n$. For $c_n < c <_g$, in which $(c_g, E(c_g))$ is a point on the energy curve at which $E(c_g) = 0$, $E(c)$ is larger than $E(0)$ and is a monotonically decreasing function of c. For $c > c_g$, $E(c)$ is less than $E(0)$ and is a monotonically decreasing function of c. So for

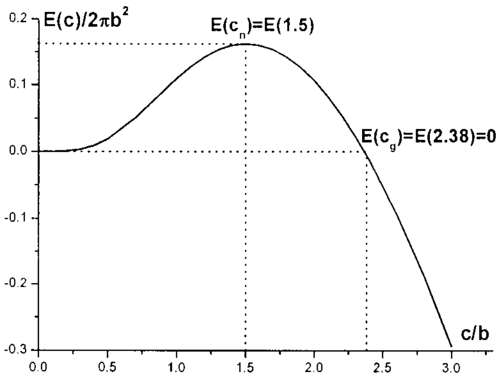

Figure 11. Energy curve for $a = 2{\cdot}0$ and $\alpha = 1{\cdot}4$.

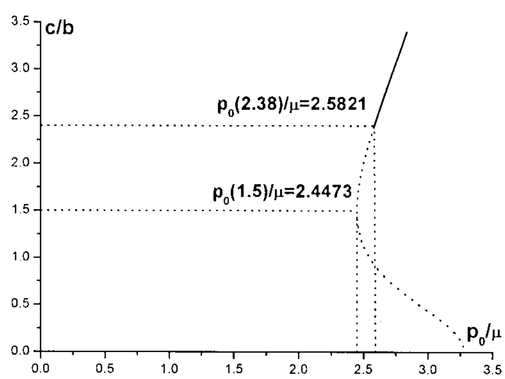

Figure 12. Stability of solutions for $a = 2{\cdot}0$ and $\alpha = 1{\cdot}4$.

$a = 2{\cdot}0$ and $\alpha = 1{\cdot}4$, the cavitation solution is stable when $c > c_g$. Thus, the cavity with radius c suddenly appears in a discontinuous fashion when $p_0 > p_g$, that is, the bifurcation is a 'snap cavitation'. The stability of the solution is shown in Figure 12 as $a = 2{\cdot}0$ and $\alpha = 1{\cdot}4$.

6. Bifurcation condition

To sum up, we have seen that the cavitated bifurcation for the cylinder may occur locally either to the right or to the left, depending on the degree of anisotropy of the material. Now we try to discuss the condition for bifurcation to occur to the right or the left following the bifurcation and energy curves.

As is shown by the bifurcation curves in Figures 3 and 4, it is possible for bifurcation to occur to the right when $p_0 > p_{\mathrm{cr}}$ and it is possible to occur to the left when $p_0 < p_{\mathrm{cr}}$ for infinite small c. As $p_0 = p_0(a)$ may be obtained from (23) and $p_{\mathrm{cr}} = p_{\mathrm{cr}}(a)$ may be obtained from (25), the critical value of a may be obtained from the equation $p_0(a) - p_{\mathrm{cr}}(a) = 0$ for infinite small c and $a_{\mathrm{cr}} = 0{\cdot}215$. Thus, we come to conclusion: the bifurcation is to the right if $a < a_{\mathrm{cr}}$ and it is to the left if $a > a_{\mathrm{cr}}$.

From the energy curves in Figures 10 and 11 and the bifurcation curves in Figures 3 and 4, the right bifurcation may occur when $E(c) < E(0)$ and the left bifurcation may occur when $E(c) > E(0)$ for infinite small c. Now $E = E(a)$ may be obtained from (27) and the critical value of a may be obtained from the equation $E(a) = E(0) = 0$ for infinite small c and the value $a_{\mathrm{cr}} = 0{\cdot}215$ is consistent with the result obtained above.

7. Growth of a pre-existing void

Consider now a hollow cylinder with inner and outer radii d and b, subjected to a prescribed uniform radial dead-load on the boundary and an axial stretch. The Equations (1–17) discussed above may be used here, provided that the inner boundary condition (11) becomes

$$\tau_{rr}(d) = 0. \tag{28}$$

All the other equations now hold on the interval $d \leq R \leq b$.

By using a similar analysis, we can get an exact analytic solution as follows:

$$p_0 = \mu \left(\frac{1}{\lambda_3} + \frac{c^2}{b^2} \right)^{1/2} \left[\int_{\left(\frac{1}{\lambda_3}+\frac{c^2}{b^2}\right)^{1/2}}^{\left(\frac{1}{\lambda_3}+\frac{c^2}{d^2}\right)^{1/2}} \frac{\lambda_3^{-\alpha} \nu^{-1-\alpha} + 6a\alpha^{-1}\lambda_3^{-6}\nu^{-7} - 6a\alpha^{-1}\lambda_3^{-2}\nu^{-3} - \nu^{\alpha-1}}{\lambda_3^{-1} - \nu^2} \mathrm{d}\nu \right]. \tag{29}$$

For a given value of p_0, we can get the corresponding constant c from (29). The solution $c > d$ describes the growth of the pre-existing void (with undeformed radius d) in the cylinder. The numerical results given by (29), describing the growth of the pre-existing voids with different values of d, are, respectively, shown in Figures 3 and 4 for $a = 0{\cdot}1$ and $\alpha = 1{\cdot}8$, as well as $a = 2{\cdot}0$ and $\alpha = 1{\cdot}4$. As shown, the radius of the pre-existing void changes slowly, even if p_0 increases when p_0 is much less than p_{cr}, but the radius experiences a sudden rapid increase when p_0 reaches a certain value (less than p_{cr}), depending on the parameters a and d/b. For $a = 0{\cdot}1$ and $\alpha = 1{\cdot}8$, the growth of the void radius is continuous, but for $a = 2{\cdot}0$ and $\alpha = 1{\cdot}4$, the growth is discontinuous and a jump may occur. Thus, the bifurcation model can be interpreted as describing sudden rapid growth of a pre-existing micro-void as was first shown in [3].

8. Effect of $f(I_5)$

We can see that $f(I_5)$ in the strain energy function W satisfying the normalization condition may have many different forms, such as $f(I_5) = I_5^2 - 2I_5 + 1$ (see [18]) or $f(I_5) = I_5^4 - 4I_5 + 3$ and so on. For different forms of $f(I_5)$, the corresponding formulae should be changed. For example, the formula (25) for the critical load in the former case of $f(I_5)$ is

$$p_{\mathrm{cr}} = \mu \sqrt{\frac{1}{\lambda_3}} \int_{\sqrt{\frac{1}{\lambda_3}}}^{\infty} \frac{\lambda_3^{-\alpha} \nu^{-1-\alpha} + 4a\alpha^{-1}\lambda_3^{-4}\nu^{-5} - 4a\alpha^{-1}\lambda_3^{-2}\nu^{-3} - \nu^{\alpha-1}}{\lambda_3^{-1} - \nu^2} \mathrm{d}\nu \tag{30}$$

The numerical results obtained from (30) are shown in Figure 1 (the scatter curves). It is clear that the results are different from each other for $a > 0$. In fact, the different form of $f(I_5)$ represents different materials and the result should be different.

9. Conclusion

The bifurcation problem of void formation and growth for incompressible, transversely isotropic hyper-elastic materials under a uniform raial tensile boundary dead-load p_0 and an axial stretch λ_3 has been studied. For all values of p_0 and λ_3, one solution corresponding to a trivial homogenous state in which the cylinder remains undeformed always exists. However, for sufficiently large values of p_0 (when p_0 is larger than its critical value), another solution involving a suddenly formed cavity bifurcates form the trivial solution. In contrast to isotropic materials, bifurcation was shown to occur locally either to the right or to the left, depending on the degree of anisotropy of the material is less than its critical value of the degree of anisotropy. When the degree of anisotropy of the material. There exists a critical value, the bifurcation is to the right and when the degree of anisotropy of the materials is larger than its critical value, the bifurcation is to the left. Through comparison of the associated potential energies, the cavitation solution is shown to be stable and the trivial solution is unstable. It has been shown that an axial stretch $\lambda_3 > 1$ can cause void formation. The phenomena of stress jumping and concentration have been discussed along with the stress distributions subsequent to the cavitation. The growth of a pre-existing void in the material for the bifurcation to the right or to the left had different characteristics. For different forms of $f(I_5)$ in anisotropic materials, the corresponding results are also different.

In conclusion, it is apparent that, despite the considerable progress what have bee obtained, many problems in connection with the modeling and analysis of cavitation-phenomena problems in hyper-elastic materials still remain to be resolved. In comparison with incompressible materials, it is more difficult to obtain an analytic solution for compressible materials, because some equations involve the incompressibility condition. Our interest is to solve the corresponding problems for compressible materials. We also intend to consider similar problems in elastodynamics for hyperelastic materials.

Acknowledgements

This work was supported by a grant form the National Nature Sience Foundation of China (19808028).

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